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A MODIFIED MULTHOPP–KALANDIYA METHOD IN THE CONTACT PROBLEM FOR A SLIDER BEARING†

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Problems of the equilibrium of an elastic circular disc and an elastic plane with a round hole under plane strain conditions in displacements are considered. Then, on the basis of the solution of these problems, the contact problem for a slider bearing is formulated. With respect to the contact pressure, an integral equation of the first kind with a difference kernel having a singularity of logarithmic form is obtained. A special version of the Multhopp–Kalandiya method is developed to solve this equation. Numerical results are given. © 2000 Elsevier Science Ltd. All rights reserved.

The contact problem for a slider bearing has repeatedly been investigated by other methods; note, for example, the monograph by Teplyi [1] and the recent paper by Kovalenko [2]. Below, unlike other researchers, for the first time a characteristic of practical importance – the relation between the impression force and the indentation – is obtained.

1. THE GENERAL SOLUTION OF THE EQUATIONS OF ELASTICITY THEORY IN A POLAR SYSTEM OF COORDINATE 3

Under conditions of plane strain, we will seek the solution of Lamé's equations when there are no mass forces in a polar system of coordinate 3 r, θ in the form

$$u_r = \sum_{n=0}^{\infty} u_r^{(n)}(r)\cos n\theta, \quad u_\theta = \sum_{n=1}^{\infty} u_\theta^{(n)}(r)\sin n\theta \tag{1.1}$$

After simple but lengthy calculations, we obtain [3]

$$\begin{aligned}
&u_r^{(0)}(r) = (1-\varkappa)C_1^{(0)}y - (1+\varkappa)C_2^{(0)}y^{-1}\\
&u_r^{(1)}(r) = 2[(2-\varkappa)C_1^{(1)}y^2 - (2+\varkappa)C_2^{(1)}y^{-2} + D_1^{(1)}(1-\varkappa\ln y) - \varkappa D_2^{(1)}]\\
&u_\theta^{(1)}(r) = -2[(2+\varkappa)C_1^{(1)}y^2 + (2+\varkappa)C_2^{(1)}y^{-2} - \varkappa(D_1^{(1)}\ln y + D_2^{(1)})]\\
&\begin{Bmatrix} u_r^{(n)}(r) \\ u_\theta^{(n)}(r) \end{Bmatrix} = \pm 2[(n+1\mp\varkappa)C_1^{(n)}y^{n+1} \mp (n+1+\varkappa)C_2^{(n)}y^{-n-1} +\\
&+(n-1-\varkappa)D_1^{(n)}y^{n-1} \mp (n-1\pm\varkappa)D_2^{(n)}y^{-n+1}] \quad (n\geqslant 2)
\end{aligned} \tag{1.2}$$

where $y = r/a$, a is the characteristic radius of the body, $C_1^{(n)}$, $C_2^{(n)}$, $D_1^{(n)}$ and $D_2^{(n)}$ are arbitrary constants, $\varkappa = 3 - 4\nu$ and ν is Poisson's ratio.

From the formulae of Hooke's law we determine the stresses

$$\sigma_r = \frac{2G}{r}\sum_{n=0}^{\infty}\sigma_r^{(n)}(r)\cos n\theta, \quad \sigma_\theta = \frac{2G}{r}\sum_{n=0}^{\infty}\sigma_\theta^{(n)}(r)\cos n\theta$$

$$\tau_{r\theta} = \frac{2G}{r}\sum_{n=1}^{\infty}\tau_{r\theta}^{(n)}(r)\sin n\theta$$

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$$
\begin{aligned}
&\left.\begin{matrix}\sigma_r^{(0)}(r)\\ \sigma_\theta^{(0)}(r)\end{matrix}\right\} = -2C_1^{(0)}y \pm (1+\varkappa)C_2^{(0)}y^{-1}\\
&\sigma_r^{(1)}(r) = 2[-2C_1^{(1)}y^2 + 2(2+\varkappa)C_2^{(1)}y^{-2} - (3+\varkappa)D_1^{(1)}/2]\\
&\sigma_\theta^{(1)}(r) = -2[6C_1^{(1)}y^2 + 2(2+\varkappa)C_2^{(1)}y^{-2} + (1-\varkappa)D_1^{(1)}/2]\\
&\tau_{r\theta}^{(1)}(r) = -2[2C_1^{(1)}y^2 - 2(2+\varkappa)C_2^{(1)}y^{-2} + (1-\varkappa)D_1^{(1)}/2]\\
&\left.\begin{matrix}\sigma_r^{(n)}(r)\\ \sigma_\theta^{(n)}(r)\end{matrix}\right\} = \pm 2[(n+1)(n\mp 2)C_1^{(n)}y^{n+1} + (n+1)(n+1+\varkappa)C_2^{(n)}y^{-n-1} +\\
&+(n-1)(n-1-\varkappa)D_1^{(n)}y^{n-1} + (n-1)(n\pm 2)D_2^{(n)}y^{-n+1}] \quad (n \geqslant 2)\\
&\tau_{r\theta}^{(n)}(r) = -2[n(n+1)C_1^{(n)}y^{n+1} - (n+1)(n+1+\varkappa)C_2^{(n)}y^{-n-1} +\\
&+(n-1)(n-1-\varkappa)D_1^{(n)}y^{n-1} - n(n-1)D_2^{(n)}y^{-n+1}] \quad (n \geqslant 2)
\end{aligned}
\tag{1.3}
$$

where G is the shear modulus.

2. THE EQUILIBRIUM OF A CIRCULAR DISC AND A PLANE WITH A CIRCULAR HOLE UNDER THE ACTION OF AN ARBITRARY NORMAL LOAD

We will consider the problem of the equilibrium of an elastic disc of radius a loaded at its centre by a point force P, which is balanced by a normal load $q(\theta)$ distributed at $r = a$ (Fig. 1). Let $q(\theta)$ be a sufficiently smooth, even function. Then

$$q(\theta) = \frac{1}{2}q_0 + \sum_{n=1}^{\infty} q_n \cos n\theta, \quad q_n = \frac{1}{\pi}\int_{-\pi}^{\pi} q(\psi)\cos n\psi d\psi \tag{2.1}$$

The boundary conditions of the problem at $r = a$ have the form

$$\sigma_r = -q(\theta), \quad \tau_{r\theta} = 0 \tag{2.2}$$

Furthermore, we take into account the fact that

$$\lim r\int_{-\pi}^{\pi}(\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)d\theta = -P \quad (r \to 0) \tag{2.3}$$

and there is no rigid displacement of the disc in the direction of the y axis, defined by the formula

$$v = u_r\cos\theta - u_\theta\sin\theta \tag{2.4}$$

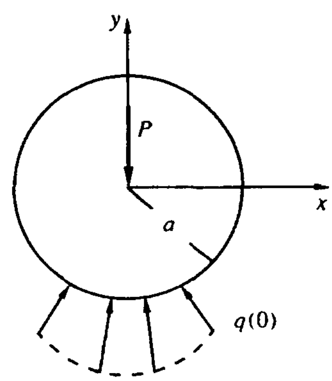

Fig. 1.

From conditions (2.2) and (2.3), on the basis of formulae (1.3), we obtain

$$C_2^{(n)}=0\,(n\geqslant 0),\quad D_2^{(n)}=0\,(n\geqslant 2)$$
$$C_1^{(0)}=\frac{q_0a}{8G},\quad C_1^{(1)}=-\frac{q_1a(1-\varkappa)}{16G(1+\varkappa)}$$
$$C_1^{(n)}=\frac{q_na}{8G(n+1)}\,(n\geqslant 2),\quad D_1^{(n)}=-\frac{q_nan}{8G(n-1)(n-1-\varkappa)}\quad (n\geqslant 2) \tag{2.5}$$
$$D_1^{(1)}=\frac{q_1a}{4G(1+\varkappa)},\quad D_1^{(1)}=\frac{P}{4G\pi(1+\varkappa)}$$

Modified Multhopp–Kalandiya method in the contact problem for a slider bearing. The last two equalities of (2.5) are not contradictory if one bears in mind that, from the condition of equilibrium of the disc

$$P=a\int_{-\pi}^{\pi}q(\theta)\cos\theta d\theta \tag{2.6}$$

it follows that $q_1=P(\pi a)^{-1}$. The condition that there are no rigid displacement of the disc in the direction of the y axis, based on the formulae (1.2), gives

$$D_2^{(1)}=0 \tag{2.7}$$

To sum up, from formulae (1.1) and (1.2), using (2.1) and the results of (2.5) and (2.7), we obtain for u_r at $r=a$ the expression

$$u_r=-\frac{a}{8\pi G}\left[-(1-\varkappa)\int_{-\pi}^{\pi}q(\psi)d\psi+\frac{(2-\varkappa)(1-\varkappa)-4}{1+\varkappa}\int_{-\pi}^{\pi}q(\psi)\cos td\psi+\right.$$
$$\left.+2\sum_{n=2}^{\infty}\left(\frac{\varkappa}{n+1}+\frac{1}{n-1}\right)\int_{-\pi}^{\pi}q(\psi)\cos ntd\psi\right]\quad(t=\theta-\psi) \tag{2.8}$$

Summing the series here, using the relations [4]

$$\sum_{n=1}^{\infty}\frac{\cos nt}{n}=-\ln\left|2\sin\frac{t}{2}\right|,\quad \sum_{n=1}^{\infty}\frac{\sin nt}{n}=\frac{1}{2}(\pi\operatorname{sgn}t-t) \tag{2.9}$$

finally, at $r=a$, we will have

$$u_r=-\frac{a}{\pi\vartheta}\int_{-\pi}^{\pi}q(\psi)K_1(t)d\psi\quad\left(\vartheta=\frac{G}{1-\nu}=\frac{E}{2(1-\nu^2)}\right)$$
$$K_1(t)=-\cos t\ln\left|2\sin\frac{t}{2}\right|-\frac{1}{2}-\frac{1-\varkappa}{2(1+\varkappa)}\sin t(\pi\operatorname{sgn}t-t)-\frac{1+2\varkappa}{(1+\varkappa)^2}\cos t \tag{2.10}$$

We will now consider the problem of the equilibrium of an elastic plane with a hole of radius a under the action of a normal load $q(\theta)$ of the form (2.1) distributed at $r=a$ (Fig. 2).

The boundary conditions of the problem at $r=a$ have the form of (2.2). Furthermore, we will take into account that, as $r\to\infty$, relation (2.3) holds and there is no rigid displacement of the plane with the hole in the direction of the y axis. Then, by the scheme set out above for the previous problem, using formulae (1.1)–(1.3) we obtain for u_r at $r=a$ the expression

$$u_r=\frac{a}{\pi\vartheta}\int_{-\pi}^{\pi}q(\psi)K_2(t)d\psi \tag{2.11}$$
$$K_2(t)=-\cos t\ln\left|2\sin\frac{t}{2}\right|+\frac{1-\varkappa}{2(1+\varkappa)}\sin t(\pi\operatorname{sgn}t-t)+\frac{1}{(1+\varkappa)^2}\cos t$$

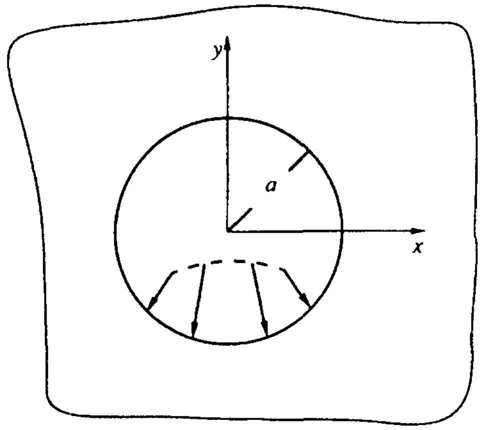

Fig. 2.

3. FORMULATION OF THE CONTACT PROBLEM FOR A SLIDER BEARING

Using relations (2.10) and (2.11), we will investigate the contact problem for a slider bearing.

Let an elastic disc of radius $b = a - \Delta$ be indented by a force P applied at its centre into the surface of a hole of radius a in an elastic plane. For simplicity we will assume below that the disc and the plane are made of the same material and that $\Delta/a \ll 1$.

Under the action of the force P, the disc moves progressively in the direction of the force by an amount δ and, between the surfaces of the disc and the plane with the hole, a contact region $|\theta| \leqslant \alpha$ is formed (Fig. 3 – for clarity, the quantities Δ and δ are exaggerated in the figure). In the contact region a contact pressure $p(\theta)$ arises, balancing the force P. The condition of contact of the disc and the plane with the hole, taking into account the difference Δ in the radii of the disc and the hole and the rigid displacement of the disc in the direction of the force by an amount δ, can be written in the form [5]

$$-u_1(\theta) + u_2(\theta) = (\Delta + \delta)\cos\theta - \Delta \tag{3.1}$$

where $u_1(\theta)$ and $u_2(\theta)$ are the radial displacements in the region of contact of points of the disc and the plane with the hole, respectively.

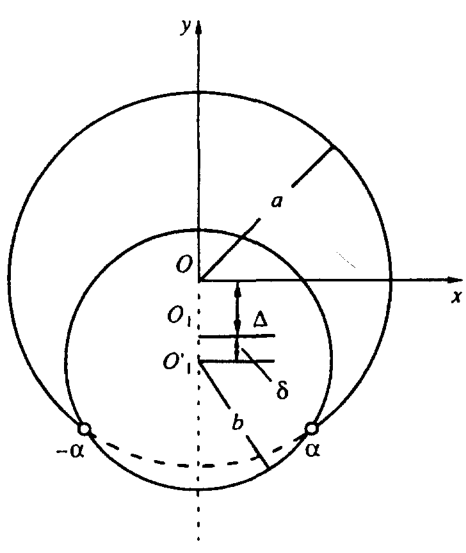

Fig. 3.

Adopting expression (2.10) for $u_1(\theta)$ (here, in view of the smallness of Δ/a, we assume that $b = a$) and expression (2.11) for $u_2(\theta)$, and assuming in (2.10) and (2.11) that

$$q(\theta) = p(\theta) \quad (|\theta| \leq \alpha), \quad q(\theta) = 0 \quad (\alpha < |\theta| \leq \pi) \tag{3.2}$$

from (3.1) we obtain the integral equation

$$-\int_{-\alpha}^{\alpha} p(\psi) \ln\left|2\sin\frac{t}{2}\right| d\psi = \frac{\pi\vartheta}{2a}\delta(\theta) - \int_{-\alpha}^{\alpha} p(\psi)F(t)d\psi \quad (|\theta| \leq \alpha) \tag{3.3}$$

$$\delta(\theta) = (\Delta + \delta)\cos\theta - \Delta$$

$$F(t) = (1-\cos t)\ln\left|2\sin\frac{t}{2}\right| - \frac{\varkappa}{(1+\varkappa)^2}\cos t - \frac{1}{4}$$

which can be used to determine the contact pressure. In order to complete the formulation of the problem, it is necessary to add to Eq. (3.3) the condition of equilibrium of the disc

$$P = a\int_{-\alpha}^{\alpha} p(\theta)\cos\theta d\theta \tag{3.4}$$

and the condition that the contact pressure at the points $\theta = \pm\alpha$ is limited, which is usually reduced to the form [6]

$$p(\pm\alpha) = 0 \tag{3.5}$$

4. METHOD OF SOLUTION

Modifying the Multhopp–Kalandiya method [7–9], we will use it to solve integral equation (3.3).

Using the known results in [6] concerning the structure of the solution of the integral equations of contact problems, we will seek a solution of Eq. (3.3) in the form

$$p(\theta) = \frac{\cos(\theta/2)\omega(\theta)}{\sqrt{2[\sin^2(\alpha/2) - \sin^2(\theta/2)]}} \tag{4.1}$$

Substituting expression (4.1) into (3.3) and replacing the variables according to the formulae

$$x = \frac{\sin(\theta/2)}{\sin(\alpha/2)}, \quad \xi = \frac{\sin(\psi/2)}{\sin(\alpha/2)} \tag{4.2}$$

after some reduction we rewrite integral equation (3.3) as follows:

$$-\int_{-1}^{1} \frac{\omega_*(\xi)}{\sqrt{1-\xi^2}}[\ln|x-\xi| + L_\alpha]d\xi = \frac{\pi\vartheta}{2\sqrt{2}a}\delta_*(x) - \int_{-1}^{1}\frac{\omega_*(\xi)}{\sqrt{1-\xi^2}}\Psi(x,\xi)d\xi \tag{4.3}$$

$$\omega_*(x) = \omega(g(x)), \quad L_\alpha = \ln[2\sin(\alpha/2)], \quad \delta_*(x) = \delta(g(x))$$

$$\Psi(x,\xi) = [F(g(x) - g(\xi)) + F(g(x) + g(\xi))]/2, \quad g(x) = 2\arcsin(x\sin(\alpha/2))$$

Again, we will change the variables in integral equation (4.3) according to the formulae

$$x = \cos\theta', \quad \xi = \cos\psi'$$

and rewrite it as follows (the primes are omitted below):

$$-\int_0^\pi \tilde{\omega}(\psi)[\ln|\cos\theta - \cos\psi| + L_\alpha]d\psi = \frac{\pi}{2\sqrt{2}}\tilde{\delta}(\theta) - \int_0^\pi \tilde{\omega}(\psi)\Psi(\cos\theta, \cos\psi)d\psi \tag{4.4}$$

$$\tilde{\omega}(\theta) = (\vartheta\delta)^{-1}a\omega_*(\cos\theta), \quad \tilde{\delta}(\theta) = \delta^{-1}\delta_*(\cos\theta)$$

Note also that, by virtue of the second formula of (3.3),

$$\tilde{\delta}(\theta) = (1+\varepsilon)[1-2\cos^2\theta\sin^2(\alpha/2)]-\varepsilon, \quad \varepsilon = \Delta/\delta$$

The following results are necessary below

A. The Lagrangian polynomial with respect to the Chebyshev nodes

$$\theta_n = \pi(2n-1)[4(i+1)]^{-1}, \quad n = 1, 2, \ldots, i+1; \quad i \geqslant 1$$

for an even function $f(\theta)(0 \leqslant \theta \leqslant \pi)$ is given by the formula [7–9]

$$f(\theta) \approx \frac{1}{i+1}\sum_{n=1}^{i+1} f(\theta_n)[1+2\sum_{m=1}^{i}\cos(2m\theta_n)\cos(2m\theta)] \tag{4.5}$$

B. On the basis of (4.5), we have a Gauss-type quadrature formula [7–9].

$$\int_0^\pi f(\psi)d\psi = \frac{\pi}{i+1}\sum_{n=1}^{i+1} f(\theta_n) \tag{4.6}$$

C. For the integral operator on the left-hand side of Eq. (4.4), the following value relation holds [10]

$$-\int_0^\pi \cos(2m\psi)[\ln|\cos\theta-\cos\psi|+L_\alpha]d\psi = \begin{cases} -\pi\ln[\sin(\alpha/2)] & (m=0) \\ \pi(2m)^{-1}\cos(2m\theta) & (m \geqslant 1) \end{cases} \tag{4.7}$$

We substitute the function $\tilde{\omega}(\theta)$ in the form (4.5) into the left-hand side of integral equation (4.4) and evaluate the integral by means of relation (4.7). On the right-hand side of Eq. (4.4), we evaluate the integral using formula (4.6). After the integrals have been evaluated, we assume in the relation obtained from (4.4) that $\theta = \theta_k$ ($k = 1, 2, \ldots, i+1$). As a result, we arrive at the following system of linear algebraic equations for $\tilde{\omega}(\theta_n)$

$$\sum_{n=1}^{i+1}\tilde{\omega}(\theta_n)\{-\ln[\sin(\alpha/2)]+\Phi_i(\theta_k,\theta_n)+\Psi(\cos\theta_k,\cos\theta_n)\} = \frac{i+1}{2\sqrt{2}}\tilde{\delta}(\theta_k) \tag{4.8}$$

$$\Phi_i(\theta,\psi) = \sum_{m=1}^{i}\frac{1}{m}\cos(2m\theta)\cos(2m\psi)$$

5. RESULTS OF THE SOLUTION

After solving system of equations (4.8), the required contact-pressure function $p(\theta)$, taking into account relations (4.1) and (4.3)–(4.5), can be represented in the form

$$p(\theta) \approx \frac{\vartheta\delta\cos(\theta/2)}{a\sqrt{2}\sin(\alpha/2)\sqrt{1-x^2}}\sum_{m=0}^{i}a_m T_{2m}(x) \tag{5.1}$$

$$a_0 = \frac{1}{i+1}\sum_{n=1}^{i+1}\tilde{\omega}(\theta_n), \quad a_m = \frac{2}{i+1}\sum_{n=1}^{i+1}\tilde{\omega}(\theta_n)\cos(2m\theta_n) \quad (m \geqslant 1)$$

where x is given by formula (4.2) and $T_{2m}(x)$ are Chebyshev polynomials of the first kind. From formula (3.4) it is now easy to obtain a relation Between the force P acting on the disc with the displacement of the dis δ in the direction of the force

$$P = -\vartheta\delta\pi\sqrt{2}f_1(\alpha) \tag{5.2}$$

$$f_1(\alpha) = a_0[\sin^2(\alpha/2)-1]+a_1\sin^2(\alpha/2)/2$$

Using the formula

$$T_{2m}(x)-1=-2(1-x^2)\sum_{s=0}^{m-1}U_{2s}(x) \quad (m\geq 1) \tag{5.3}$$

where $U_{2s}(x)$ are Chebyshev polynomials of the second kind, and the boundedness condition (3.5), we rewrite the expression for the contact pressure (5.1), taking account of (5.2), in the form

$$p(\theta)=P\frac{\sqrt{(1-x^2)[1-x^2\sin^2(\alpha/2)]}}{\pi a f_1(\alpha)\sin(\alpha/2)}\sum_{m=1}^{i}a_m\sum_{s=0}^{m-1}U_{2s}(x) \tag{5.4}$$

The boundedness condition for the contact pressure (3.5) in this case will itself take the form

$$\sum_{m=0}^{i}a_m=0 \tag{5.5}$$

from which the relation between ε and α can be found numerically. Let this relation be given by $\varepsilon=f_2(\alpha)$; then, by virtue of (5.2), we will also have

$$P=-\vartheta\Delta\pi\sqrt{2}f_1(\alpha)/f_2(\alpha) \tag{5.6}$$

We will find the maximum contact pressure intensity factor introduced in [1], by the formulae

$$K=p(0)/p, \quad p=P/(\pi a) \tag{5.7}$$

From (5.4) we obtain

$$K=\frac{f_3(\alpha)}{f_1(\alpha)\sin(\alpha/2)}, \quad f_3(\alpha)=\sum_{m=1}^{k}a_{2m-1} \tag{5.8}$$

where $i=2k$ (even) or $i=2k-1$ (odd).

We will carry out specific calculations with $\nu=0.3$. Figure 4 shows graphs of K (curve 1), $P/(E\delta)$ (curve 2) and $P/(E\Delta)$(curve 3) against α. Curves 1 and 3 agree excellently with the results obtained by another method ([1, p. 22]). Curve 2, which is of practical importance, has been obtained for the first time. Curve 3, for specified P and Δ, is used to determine the angle of contact α, and, for a specified value of P and an already known α, the indentation δ can then be found from curve 2.

Figure 5 shows graphs of the ratio $p(\theta)/p$ as a function of x, where curves 1–5 correspond to $\alpha=\pi/18, \pi/9, \pi/6, 2\pi/9$ and $\pi/3$. All calculations were carried out with $i=14$ to 16 significant digits. If a smaller number of digits is returned, the error rapidly builds up at fairly large angles $\alpha<\pi$.

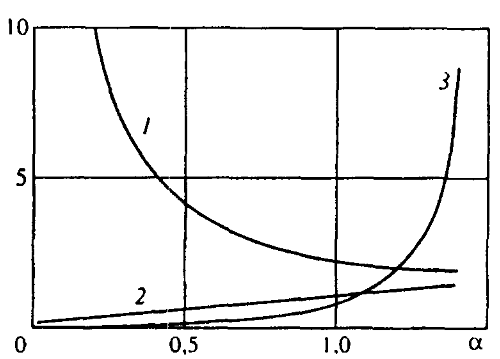

Fig. 4.

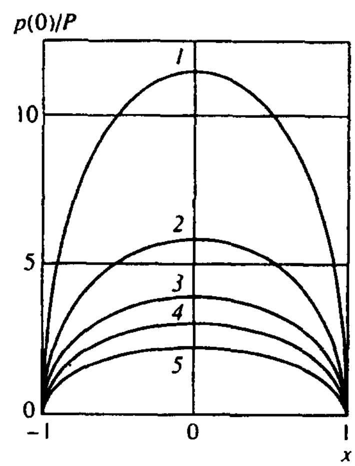

Fig. 5.

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